UNIVERSITÉ DU LUXEMBOURG
ANALYSE 1 POUR MATHÉMATICIENS
2015-2016

EXERCISE SHEET 2

2.1. Compute the following limits:

2.1.1. $\lim_{n\to+\infty} \frac{\sqrt{n^2+2}}{2n}$

2.1.2. $\lim_{n\to+\infty} \frac{\cos\sqrt{n}}{n}$

2.1.3. $\lim_{n\to+\infty} n^2 \cos(\frac{1}{n^2}) \sin(\frac{1}{n^3})$

2.1.4. $\lim_{n\to+\infty} \frac{\sin(n+1)-\sin(n-1)}{\cos(n+1)+\cos(n-1)}$

2.1.5. $\lim_{n\to+\infty} \frac{\sin\sqrt{n^3+n^2+1}}{n^3+n^2+1}$

2.1.6. $\lim_{n\to+\infty} \frac{n(\sqrt{n^2+1}-\sqrt{n^2+4})}{2}$

2.1.7. $\lim_{n\to+\infty} n(\sqrt{n^4+4n+5}-n^2)$

2.2. Given a positive number $a \in \mathbb{R}$, discuss the convergence of the sequence $x_n = a^n$.

2.3. Given a positive number $a \in \mathbb{R}$, discuss the convergence of the sequence $x_n = a^{\frac{1}{n}}$.

2.4. Prove the following criteria for the convergence of a sequence:

2.4.1. Let $\{a_n\}_n$ be a sequence of positive real numbers. Suppose that there exists

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \ell \ .$$

Prove that:

- if $\ell > 1$, then the sequence a_n is divergent;
- if $\ell < 1$, then the sequence $a_n \to 0$;

Show with an example that if $\ell = 1$, we can't determine the behaviour of the sequence a_n.

2.4.2. Let $\{a_n\}_n$ be a sequence of positive real numbers. Suppose that there exists

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell \ .$$

Prove that:

- if $\ell > 1$, then the sequence a_n is divergent;
- if $\ell < 1$, then the sequence $a_n \to 0$;

Show with an example that if $\ell = 1$, we can't determine the behaviour of the sequence a_n.

2.4.3. Let $\{a_n\}_n$ be a sequence of positive real numbers. Suppose that there exists

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell \ .$$

Then $\lim_{n\to\infty} \sqrt[n]{a_n} = \ell$.

2.5. Compute the following limits (with the aid of the above criteria)

2.5.1. $\lim_{n\to\infty} \frac{a^n}{n^b}$ for every $a > 1$ and $b > 0$.

2.5.2. $\lim_{n\to\infty} \frac{a^n}{n!}$ for every $a > 1$.

2.5.3. $\lim_{n\to\infty} \frac{n^n}{n!}$.

2.5.4. $\lim_{n\to\infty} \sqrt[n]{a}$ for every $a > 0$.

2.5.5. $\lim_{n\to\infty} \sqrt[n]{n}$.

2.5.6. $\lim_{n\to\infty} (n^3 + 7n^2 + 5)^{\frac{1}{n}}$.

2.5.7. $\lim_{n\to\infty} \sqrt[n]{n!}$.

2.5.8. $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$.

2.5.9. $\lim_{n\to\infty} \sqrt[n]{\binom{4n}{n}}$.

2.6. Compute the infimum and the supremum of the following sets:

2.6.1. $A = \{\frac{n}{mn+1} \mid m, n \in \mathbb{N}^*\}$

2.6.2. $B = \{\frac{1}{n} + (-1)^n \mid n \in \mathbb{N}^*\}$

2.6.3. $C = \{x \in \mathbb{R} \mid \exists (m, n) \in \mathbb{N}^* \times \mathbb{N}^* : x = \frac{1}{m} + \frac{1}{n}\}$

2.7. Let $\{x_x\}_n$ be a convergent sequence and let $\{y_n\}_n$ be the sequence defined by

$$y_n = x_{n+1} - x_n \ .$$

Prove that the sequence $\{y_n\}_n$ converges and compute its limit.
Determine what happens when the sequence $\{x_n\}_n$ is divergent. In particular find an example such that $\{y_n\}_n$ converges to 0.

2.8. Let $\{a_n\}_n$ be a sequence of real numbers. Consider its arithmetic mean

$$s_n = \frac{a_1 + \dots a_n}{n}$$

- Prove that if $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} s_n = 0$.
- More in general, prove that if $\lim_{n\to\infty} a_n = \ell$, then $\lim_{n\to\infty} s_n = \ell$.
- Find an example such that $\{a_n\}_n$ diverges but $\{s_n\}_n$ converges.
- Compute the limit of the sequence $\{x_n\}_n$ defined by

$$x_n = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k}$$

2.9. ****.** Given a sequence of real numbers $\{x_n\}_n$ we define

$$\limsup_{n\to\infty} x_n = \inf_k \sup_{n\geq k} a_k$$

$$\liminf_{n\to\infty} x_n = \sup_k \inf_{n\geq k} a_k \ .$$

- Prove the existence of $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.
- Prove that $\limsup_{n\to\infty} x_n \geq \liminf_{n\to\infty} x_n$ and the equality holds if and only if $\{x_n\}_n$ converges.
- Compute $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ when $x_n = \sin(n)$.

2.10. **. Let $\{a_n\}_n$ and $\{b_n\}_n$ sequences of real numbers such that

- $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$
- $\{b_n\}_n$ is monotone decreasing.

Prove that

$$\liminf_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} \le \liminf_{n\to\infty} \frac{a_n}{b_n} \le \limsup_{n\to\infty} \frac{a_n}{b_n} \le \limsup_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}.$$